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Non-classical symmetries and the singular manifold method revisited

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Abstract. The connection between the singular manifold method (Painlevé expansions truncated at the constant term) and symmetry reductions of two members of a family of Cahn–Hilliard equations is considered. The conjecture that similarity information for a nonlinear partial differential equation may always be fully recovered from the singular manifold method is violated for these equations, and is thus shown to be invalid in general. Given that several earlier examples demonstrate the connection between the two techniques in some cases, it now becomes necessary to establish when such a relationship exists—a question related to a deeper understanding of Painlevé analysis. This issue is also briefly discussed.

1. Introduction

In the search for a complete characterization of the *integrability* of nonlinear partial differential equations (NLPDEs) attempts have been made to relate different properties of soliton equations, such as the underlying Lie algebraic properties, the Hirota bilinearization technique, and the Painlevé property [1, 2]. A more recent approach [3–6] has been to broaden the scope to include non-integrable NLPDEs as well. In particular, these recent efforts have recovered the symmetries of NLPDEs (obtained by either the method of non-classical symmetries [7–9] or the less general technique proposed by Clarkson and Kruskal [10–12]) from the apparently unrelated technique of singular manifold expansions [1, 2, 13, 14] (or Painlevé expansions truncated at the constant term). This connection between the symmetry reductions and singular manifold expansions of NLPDEs has been persuasively demonstrated (though not proved, as the authors of [3–6] mention) by considering a large variety of NLPDEs. In all the cases considered to date, it is found that the symmetry information may be completely recovered from the singular manifold method and it has been conjectured that this may be the case for any NLPDE.

The purpose of the present paper is to consider this connection further in the context of a family of Cahn–Hilliard equations. The Painlevé analysis [15] of this family of equations, as well as their symmetry reductions obtained using both the Lie group and the Clarkson–Kruskal direct technique [16] have been considered earlier. Here, we reconsider the results of [15] and [16] more closely using the technique of [3–6] to explore the connection between them. Somewhat to our surprise, and in contrast to earlier examples, we found that the singular manifold method is unable to recover the symmetry information for these equations. In the absence of any earlier proof that non-classical symmetries and the results

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of the singular manifold method are necessarily connected, this also serves to establish that such a relationship need not exist. We shall comment further on this in section 5.

The rest of this paper is organized as follows. Section 2 briefly reviews the singular manifold method (we shall follow [3–5], and abbreviate this henceforth as SMM). Sections 3 and 4 consider the connection between the SMM and the symmetry reductions for two different members of the family of Cahn–Hilliard equations. In section 5, we present our conclusions and discuss the results. The reader should also note that some statements in [16] regarding the direct technique of Clarkson–Kruskal yielding the most general similarity reductions of a NLPDE (following the claims in [10–12]) are now dated, having been subsequently disproved [17, 18].

2. The singular manifold method

First, we briefly summarize the method. Further details may be found in [5, 13, 14]. The singular manifold method (SMM) is based on the self-consistently truncated series solutions of partial differential equations satisfying the Painlevé property (PP), or the conditional Painlevé property (CPP), in the form of

$$u = \sum_{j=0}^{\alpha} u_j(x,t)[\phi(x,t)]^{j-\alpha} = u_0\phi^{\alpha} + u_1\phi^{1-\alpha} + \cdots + u_{\alpha}. \tag{2.1}$$

Here, ϕ is the singular manifold, which makes (2.1) hold, and $u_j(x,t)$ are analytic functions that are determined in terms of ϕ through the recurrence relations obtained by substitution of (2.1) into the corresponding partial differential equation (PDE).

We also introduce the following quantities, which are used throughout the paper:

$$c = \frac{\phi_t}{\phi_x} \tag{2.2}$$

$$v = \frac{\phi_{xx}}{\phi_x} \tag{2.3}$$

$$s = v_x - \frac{v^2}{2}. \tag{2.4}$$

From the compatibility conditions $\phi_{xt} = \phi_{tx}$ and $\phi_{xxt} = \phi_{xtx}$, one obtains the following relationships among the quantities, c, v, and s:

$$v_t = (c_x + vc)_x \tag{2.5}$$

$$s_t = c_{xxx} + 2sc_x + cs_x. \tag{2.6}$$

The PP (CPP) are invariant under homographic transformations of ϕ [5, 19, 20]

$$\phi \to \frac{a\phi + b}{c\phi + d}. \tag{2.7}$$

Here, c and s are invariant under the Möbius or homographic group (2.7), but not v. Substituting (2.1) into the corresponding PDE, one can obtain the coefficients u_j in terms of c, s, and v. Also, it follows from the substitution that u_{α} is a solution of the PDE, which means that the ansatz (2.1) plays the additional role of being an auto-Bäcklund transformation among two solutions u and u_{α} of the same PDE. Since u_{α} is a solution of the PDE, the singular manifold must satisfy an additional equation, which can always be written as a relationship among the homographic invariants and their derivatives [5]. This last equation, together with (2.5) and (2.6), defines the singular manifold. They are called the singular manifold equations henceforth. Using this procedure we can express the solution u_α in terms of the singular manifold quantities as

$$u_\alpha = u_\alpha(v, c, s) \tag{2.8}$$

where c and s must satisfy the singular manifold equations. Clearly, the SMM is applicable to equations having both the Painlevé property (PP) or the conditional Painlevé property (CPP).

3. Cahn–Hilliard equation for $m = 1$ and one spatial variable

In this section and the next, we shall apply the singular manifold method to the Cahn–Hilliard equation with one spatial variable for the cases $m = 1$ and 2, respectively. Following the procedure outlined by Estevez and co-workers [3–5], we shall then consider the connection between the SMM and non-classical Lie group symmetries to see whether the latter are derivable from the former.

The family of Cahn–Hilliard equations considered is

$$u_t = [u^m u_x]_x - k u_{xxxx}$$

with different values of m for various cases of interest.

3.1. The singular manifold method

The Painlevé analysis for the Cahn–Hilliard equations for $m = 1$

$$u_t - u_x^2 - u u_{xx} + k u_{xxxx} = 0 \tag{3.1}$$

has already been carried out [15]. This equation possesses the CPP. The SMM allows us to write a truncated solution u' for (3.1). Let

$$u' = u_0 \phi^{-2} + u_1 \phi^{-1} + u. \tag{3.2}$$

Substitution of (3.2) into (3.1) yields a set of coupled partial differential equations order by order in powers of ϕ from $\mathrm{O}(\phi^{-6})$ to $\mathrm{O}(\phi^0)$. Solving the first two yields

$$u_0 = 12k\,\phi_x^2 \tag{3.3}$$

$$u_1 = -12k\,\phi_{xx}. \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), u' can be written as

$$u' = u - 12k \left(\frac{\phi_x}{\phi}\right)_x \tag{3.5}$$

where u must be another solution of (3.1) and ϕ is the singular manifold. Substituting (3.3) and (3.4) into the equations at the following four orders $\mathrm{O}(\phi^{-4})$ to $\mathrm{O}(\phi^0)$, and using the quantities v, s, and c defined in (2.2)–(2.4) yields the set of equations

$$u = 4ks + 3kv^2 \tag{3.6}$$

$$c + ks_x = 0 \tag{3.7}$$

$$2(c_x + ks_{xx}) + \frac{3\phi_{xx}}{\phi_x}(c + ks_x) = 0 \tag{3.8}$$

$$c_{xx} + 2V(c_x + ks_{xx}) + ks_{xxx} + \left(s + \frac{3V^2}{2}\right)(c + ks_x) = 0. \tag{3.9}$$

Using (3.7), (3.8) and (3.9), we have

$$c_x + ks_{xx} = 0 \tag{3.10}$$

$$c_{xx} + ks_{xxx} = 0. \tag{3.11}$$

Compatibility of c and s yields

$$s_t = c_{xxx} + 2c_x s + cs_x \tag{3.12}$$

which, with (3.7), yields

$$s_t + ks_{xxxx} + 2kss_{xx} + ks_x^2 = 0. \tag{3.13}$$

Using (3.6) and (3.7), the equation at $\mathrm{O}(\phi^0)$ yields

$$s_t + ks_{xxxx} + 2kss_{xx} + \frac{ks_x^2}{2} = 0. \tag{3.14}$$

Thus, from (3.13) and (3.14), the singular manifold ϕ is overdetermined and

$$s_x = 0. \tag{3.15}$$

Hence, using (3.7),

$$c = 0. \tag{3.16}$$

Also, by either (3.13) or (3.14),

$$s_t = 0. \tag{3.17}$$

Taken together, (3.15) and (3.17) imply that $s =$ constant.

Following Estevez and coworkers [3–5], we shall eliminate v from the equations because it is not a homographic invariant. From (3.6)

$$v^2 = \frac{1}{3}\left(\frac{u}{k} - 4s\right) \tag{3.18}$$

$$v_x = \frac{s}{3} + \frac{u}{6k}. \tag{3.19}$$

We also need the quantities u_x and u_t, which with the help of (2.5), (2.6), (3.15), and (3.17)–(3.19), are found to be

$$u_x = 2kvs + uv \tag{3.20}$$

$$u_t = 2uc_x - 8ksc_x + v[6kc_{xx} + 2kcs + uc]. \tag{3.21}$$

3.2. The SMM and the non-classical method

To establish the relationship between the SMM and the non-classical symmetries, the symmetry condition requires that the vector field components $\{\xi(x,t,u); \tau(x,t,u); \eta(x,t,u)\}$ satisfy the invariant surface condition [7–9]

$$\xi(x,t,u)u_x + \tau(x,t,u)u_t = \eta(x,t,u). \tag{3.22}$$

Comparing (3.20) and (3.21) with (3.22) shows that ξ and τ must be such that the result of substituting (3.20) and (3.21) into (3.22) should only be dependent on the homographic invariants c and s. Suppressing the v terms requires

$$\xi(2ks + u) + \tau(6kc_{xx} + 2kcs + uc) = 0 \tag{3.23}$$

or equivalently

$$\xi + c\tau = 0 \tag{3.24}$$

$$c_{xx} = 0. \tag{3.25}$$

Equation (3.25) is satisfied automatically since $c = 0$. Hence, from (3.24), we get $\xi = 0$ and τ arbitrary.

Putting the above results together, the generators of the infinitesimal Lie group are

$$\xi = 0 \tag{3.26}$$

$$\tau = h(x, t, u) \tag{3.27}$$

$$\eta = 0 \tag{3.28}$$

where h is an arbitrary function.

Unlike the fairly large number of NLPDEs considered in [3–6], these symmetries are not the same as those obtained from the classical and non-classical Lie group methods. The general Lie symmetry group (which in this case are just classical translations and dilations (scalings)) is [16]

$$\xi(x, t) = \alpha x + \beta \tag{3.29}$$

$$\tau(x, t) = 4\alpha t + \gamma \tag{3.30}$$

$$\eta(x, t, u) = -2\alpha u \tag{3.31}$$

where α, β, and γ are arbitrary coefficients.

4. Cahn–Hilliard equation for $m = 2$ with one spatial variable

In this section, we consider the relation between the SMM and the method of non-classical symmetries for the $m = 2$ Cahn–Hilliard equation.

4.1. The singular manifold method

The Painlevé property for the Cahn–Hilliard equations for $m = 2$

$$u_t - 2uu_x^2 - u^2u_{xx} + ku_{xxxx} = 0 \tag{4.1}$$

has been studied in [15]. The equation possesses the conditional Painlevé property (CPP). The truncated Painlevé expansion of (4.1) can be defined through the auto-Bäcklund transformation:

$$u' = u_0\phi^{-1} + u. \tag{4.2}$$

Substitution of (4.2) into (4.1) yields a set of coupled partial differential equations order by order in powers of ϕ from $O(\phi^{-5})$ to $O(\phi^0)$. Solving the first of these yields

$$u_0 = \sqrt{6k}\phi_x. \tag{4.3}$$

Thus, we have the auto-Bäcklund transformation

$$u' = \sqrt{6k}\frac{\phi_x}{\phi} + u \tag{4.4}$$

where u must be another solution of (4.1). Substituting (4.3) into the equations at $O(\phi^{-4})$ to $O(\phi^0)$ and using the quantities v, s, and c yields the set of equations

$$u = -\frac{\sqrt{6k}}{2}v \tag{4.5}$$

$$s = 0 \tag{4.6}$$

$$2ks_x + 3kvs + c = 0 \tag{4.7}$$

$$2ks_{xx} + 4kvs_x + 2c_x + 2ks^2 + 3kv^2s + 2vc = 0 \tag{4.8}$$

$$2v(c_x + cv) - 2(c_{xx} + 2vc_x + cs + \tfrac{3}{2}cv^2) + 10kv^3s + 7kv^2s_x \\ -2kvs^2 - 6kv^3s - 6kss_x - 9v^2s_x - 2kvs_{xx} - 2ks_{xxx} = 0. \tag{4.9}$$

By (4.6),

$$c = 0. \tag{4.10}$$

Thus, (4.8) and (4.9) become identities. Using (4.5), we have

$$v = -\frac{2}{\sqrt{6k}}u. \tag{4.11}$$

Next, (2.4), (4.6), and (4.11) yield

$$v_x = \frac{1}{3k}u^2. \tag{4.12}$$

We also need the quantities u_x and u_t. Using (2.5), (4.10), and (4.12), these are

$$u_x = -\sqrt{\frac{1}{6k}}u^2 \tag{4.13}$$

$$u_t = 0. \tag{4.14}$$

4.2. The SMM and the non-classical method

As in section 3.2, we next consider the connection of the SMM to the method of non-classical symmetries. The substitution of (4.13) and (4.14) into the surface invariant condition

$$\xi(x, t, u)u_x + \tau(x, t, u)u_t = \eta(x, t, u)$$

leads to

$$\eta(x, t, u) = -\xi\sqrt{\frac{1}{6k}}u^2. \tag{4.15}$$

Thus, the generators of the infinitesimal Lie group are

$$\xi = f(x, t, u) \tag{4.16}$$

$$\tau = g(x, t, u) \tag{4.17}$$

$$\eta(x, t, u) = -\xi\sqrt{\frac{1}{6k}}u^2 \tag{4.18}$$

where f and g are arbitrary functions.

For this equation the non-classical symmetries with the SMM are, once again, not the same as those determined through the classical and non-classical Lie methods. The general Lie group (which in this case are just classical translations and dilations (scalings)) generators are [16]

$$\xi(x, t) = \alpha x + \beta \tag{4.19}$$

$$\tau(x, t) = 4\alpha t + \gamma \tag{4.20}$$

$$\eta(x, t, u) = -\alpha u \tag{4.21}$$

where α, β, and γ are arbitrary coefficients. Again, this is in contrast to the various NLPDEs considered in [3–6].

5. Conclusions and discussion

The Cahn–Hilliard equation with $m = 1$ and 2 considered in sections 3 and 4 provides examples of NLPDEs for which the SMM is not versatile or general enough to recover all information regarding the symmetry properties of the equation. As mentioned, this is in contrast to the large number of examples considered in [3–6] where the relationship between the SMM and the symmetry reductions has been examined and found to hold. However, as conjectured in [3–6], these examples only served to indicate that there might indeed always be such a connection. To date, there does not exist any proof that the SMM is indeed always guaranteed to recover all symmetry reductions of any NLPDE having either the Painlevé or conditionally Painlevé property. As such, the examples in sections 3 and 4 serve to establish the converse, i.e. one cannot always recover symmetry information for an arbitrary NLPDE from the SMM.

Of course, such a connection does indeed exist for some NLPDEs, as the examples in [3–6] clearly demonstrate. However, it now becomes necessary to establish when this is indeed the case. Clearly, this is related to the difficult issue of a complete understanding of Painlevé analysis. While it is not the purpose of this paper to address this question, we should like to conclude by making some preliminary observations concerning features which might make such a connection hard to establish, or unlikely, or even non-existent. One requirement for the implementation of the technique proposed in [3–5] appears to be that after substitution of u_x and u_t in the surface invariant condition, one should be able to solve for the homographic invariant c (in terms of u and η) for use in the compatibility condition (2.6). If this is hard, then the implementation of the method of [3–5] to establish the connection between the SMM and the symmetries is difficult to accomplish. This makes the establishment of such a connection non-obvious using the technique of [3–5]. However, whether it also indicates the non-existence of such a connection for the given NLPDE, or simply points to the need for an alternative technique remains to be investigated. It appears that the choices of NLPDEs in [3–6] fortuitously resulted in SMM equations where the technique could be implemented. However, examples of the opposite are not hard to find. For instance, for the ϕ^4 equation [21], the equations for u_x and u_t (following the procedure in section 3) are

$$u_x = \left(\frac{2}{1-c^2}\right)\left[\sqrt{\frac{1-c^2}{2}}\frac{u^2}{2} + \left(\frac{1-c^2}{2}\right)^{3/2} s - ucc_x\right]$$

$$u_t = -\left(\frac{1-c^2}{2}\right)^{-1} ucc_t + \left(\frac{1-c^2}{2}\right)^{1/2} c_{xx} + uc_x + \left(\frac{1-c^2}{2}\right)^{1/2} cs + \frac{c}{2}\left(\frac{1-c^2}{2}\right)^{-1/2} u^2.$$

Solving for c (for substitution in (2.6)) in terms of u and η after substitution of these in the surface invariant condition is virtually impossible in this case. For the long-wave equations [22], the equations are even more complex and implementation of the technique of [3–5] is even harder!

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